# The Algorithm

1. Add a new node
2. Add a new node
3. Add a directed edge with capacity
4. Add a directed edge with capacity
5. Run Ford-Fulkerson on this new graph with source and sink
6. Let equal the maximum flow from that
7. Run Ford-Fulkerson on this new graph with source and sink
8. Let equal the maximum flow from that
9. Remove and and their outgoing edges
10. Add new node
11. Add a directed edge with capacity
12. add a new node
13. direct one edge from each of the two old sinks to
14. recall that
15. let (e.g. when the numerator 6, when the numerator is 7)
16. let
17. make the capacities of each of the two new directed edges we made in step 3
18. run the Ford-Fulkerson algorithm on this new graph with source and sink .
19. Let and equal the flows on edges and , respectively. They are the fair flows.

# Proof of Correctness

Observe that the most flow that could go into a sink is the minimum between the sum of the capacities of edges that go out from the source () and the sum of the capacities of edges that go into said sink (). We could thus restrict to this minimum to save time, which is implemented in steps 1 through 4. Also observe that for because is computed when there’s no constraints put on the problem. So, for the flows to be fair, both flows cannot exceed the smaller of the ’s by more than 1. If both of them are greater than , then the fair flows have to be upper bounded by . is defined like that because when is even, then obviously each sink cannot exceed , but if is odd, say 7, then the fair flows could be 3 and 4, which are upper bounded by . Thus, the fair flows to the two sinks have to be restricted to the three-way minimum in step 16, which is implemented in step 17. The explanation for step 11 is that the total flow that goes into cannot exceed the

# Runtime Analysis

Recall that the runtime complexity of Ford-Fulkerson is with being the number of edges and as defined in step 14. Step 5 is and step 7 is because we made the new sources’ ( and ) outgoing capacity to and . Step 18 is because we restricted the ’s outgoing capacity to . The runtime overall is thus .